

## **A Study on Prospective Teachers' Knowledge in the Domain of Multiplicative Structure**

Ing-jye Chang<sup>\*</sup>

### **ABSTRACT**

The purpose of this study is to set up an evaluation model in order to evaluate the prospective teachers' mathematical and pedagogical content knowledge in the domain of multiplicative structures. The study was conducted in two steps. Initially, 417 seniors at one teachers' college were given the written test. After completing the written test, thirty seniors, selected from eight departments in three different major groups, were interviewed. The reliability of the written test was determined as .81 (coefficient Alpha).

These mathematics knowledge profiles indicated that the prospective teachers were not ready for teaching. Their level of pedagogical understanding was unacceptably low (35% correct). The mean score (80% correct) on the test of mathematical content knowledge was better but not completely satisfactory. The findings showed that the prospective teachers were lacking in diagnostic teaching and remediation teaching, suggesting they were not able to represent appropriately their teaching methods using a wide variety of models. They were not willing to prove their formula, and applied incorrect mathematical knowledge to solve problems. Their explanations relied on procedural approaches, rather than a pedagogically oriented understanding.

There were significant differences between the different major groups in the mathematical content knowledge ( $F = 46.03$ ,  $df = (2, 409)$ ) and the pedagogical content knowledge ( $F = 21.74$ ,  $df = (2, 409)$ ). As a whole, the mathematics majors

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performed the best among all of the three groups. The education majors did better than the art majors on all the tasks except for remediation teaching.

**Key words: Prospective teacher, Teachers' knowledge, Multiplicative structure**

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### **INTRODUCTION**

#### **Motivation and Background of Research**

The quality of elementary school mathematics education in depends on a number of factors, the most significant of which is the teacher preparation. What kind of education knowledge should an elementary school mathematics teacher possess? What should he or she know? What methods should teacher educators use to assist teachers in obtaining both mathematics and pedagogical content knowledge? What should we do if we want to provide them with their teacher education and professional development? In effect, what should teacher educators do? How should we design and evaluate innovative undergraduate programs to prepare future elementary school mathematics teachers?

Teacher education is one priority of Taiwanese educational reform. In the past, all teachers in Taiwan were trained in teachers' colleges at all levels. In order to diversify the source and raise the quality of teachers, priorities of reform include assisting universities in offering teacher education courses like teachers' colleges and establishing a unified and consistent system for preservice teacher education, apprenticeship, qualification evaluation and on-the-job training for elementary and secondary school teachers.

In the wake of teacher education liberation, the elementary school mathematics teachers preparation programs need to be assessed. There has been a few researches

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regarding the evaluation of elementary school teachers' mathematics preparation. For example, Liu (1990) conducted an analysis of mathematics performance of inservice elementary school teachers. Chang (1990) conducted a survey to find out what basic concepts and skills elementary school mathematics teachers need, as a basis for the preservice and inservice training for the elementary school teachers. Leu (1994) investigated inservice elementary school teachers' relative knowledge regarding the teaching of fractions.

In addition, a few instructors in the teachers' college were able to conduct an evaluation of the mathematics content course and the mathematics methods course. Many prospective teachers, and many mathematics faculty, think of the required college-level course in mathematics for elementary teachers as an opportunity to review concepts.

To prepare teachers who are effective in helping students to learn is the same ultimate goal for all teacher education institutions. There is a fundamental belief that teachers influence learning (Ball & DcDiarmid, 1990).

The quality of teachers' contributions tends to be strongly influenced by their own knowledge, thus they need to develop a deep understanding of the content they are teaching. Surprisingly, many prospective teachers do not develop an adequate mathematics background during their training (Cramer & Lesh, 1988; Lacampagne et al., 1988; Post et al., 1993).

In matters related to assessment, an important challenge for mathematics education research focuses on clarifying what it means for teachers to construct a deep understanding of elementary mathematics ideas. Mathematics educators have repeatedly called for teaching mathematics with understanding (Brownell, 1987; Skemp, 1978; Van de Walle, 2001) and preparing teachers to teach for understanding (Carpenter, Fennema, Peterson, & Carey, 1988; Shulman & Grossman, 1988). It is time to conduct an evaluation research to assess the prospective teachers' content knowledge.

### **Research Purposes**

In the study, a model for assessing the prospective teachers' content knowledge is built up from three perspectives: mathematics, psychological, and pedagogical

perspectives. This study conducts an evaluation of the prospective teachers' mathematical and pedagogical content knowledge of multiplicative structures. Profiles of the prospective teachers' mathematics knowledge were developed, and compared with the performance between the different majors.

### **Research Questions**

The purpose of this study was to assess prospective elementary teachers' mathematical and pedagogical content knowledge of multiplicative structures. This research sought to answer the following questions:

1. What is the level of prospective elementary teachers' mathematical content knowledge of multiplicative structures?
2. What is the level of prospective elementary teachers' pedagogical content knowledge of multiplicative structures?
3. Is there a significant difference in mathematical content knowledge of multiplicative structures between the different majors?
4. Is there a significant difference in pedagogical content knowledge of multiplicative structures between the different majors?

### **What do we know about multiplicative structures?**

The study of multiplicative structures has been under way in recent years, following the 1983 work of Vergnaud. Confrey & Harel (1994) argue that the topics of multiplicative structures possess an interconnected and complexity, and suggest that "to learn about a multiplicative conceptual field, one must examine its relation to the situations in which multiplicative reasoning occurs and not view its ideas as isolated abstractions" (p. xi). In an attempt to provide a grounding for elementary educational practices with respect to the school mathematics related to these structures in, research is being conducted using both top-down and bottom-up approaches (Lamon, 1994; Kieren, 1994). On the one hand, researchers have looked at the formal mathematics itself, which is related to multiplicative structures including its theoretical mathematical & semantic analyses. In addition, researchers have examined children's knowledge in clinical interviews or in teaching experiments to determine which their knowledge forms a useful foundation upon which instruction might be built. So, broadly viewed, in addition to analyzing the mathematics and semantic structures,

research in multiplicative structures has analyzed student errors and has tried to figure out how people construct an understanding of mathematics concepts ( Smith & Confrey, 1994).

Multiplicative structures are sets of problems involving arithmetical operations and notions of multiplication, division, fractions, ratios, and similarities. On the other hand, additive structures refer to those whose operations and notions are of the additive type such as addition, subtraction, difference, intervals, and translation. Thus, there are distinctions between additive or absolute notions of change, and the multiplicative or relative interpretation. Students need both these perspectives if their mathematics thinking is to advance beyond elementary arithmetic.

Vergnaud (1983, 1988) introduced the idea of a multiplicative concept field (MCF) and identified the mathematical bundle of topics included in MCF.

[T]he conceptual field of multiplicative structures consists of all situations that can be analyzed as simple and multiple proportion problems and for which one usually need to multiply or divide... Among these concepts are linear and nonlinear functions, vector space, dimensional analysis, fraction, ratio, rate, rational number, and multiplication and division (Vergnaud, 1988, p. 14).

Confrey's (1994) notion of splitting enlarges the scope of multiplicative structures to include exponential functions. However, from Vergnaud's (1994) point of view the multiplicative structures can be viewed as: a set of situations that required multiplication, division, or combination of such operation; a set of schemes that required to deal with these situations; a set of concepts and theorems that make it possible to analyze the operations of thinking need; and a set of formulations and symbolizations.

Vergnaud (1983, 1988, 1994) has used a model based on the concept of measure space to see the main categories of multiplicative structures as consisting of building dimensional relationships in sample and multiple proportional situations and in the extension of concepts of rates and ratios to even more complex situations. It is clear that multiplicative structures can be analyzed in a way that leads to fractions, rational numbers, ratios and proportions.

A ratio is a comparative index that conveys the notion of relative magnitude. Ratios and proportional reasoning are critical to functioning in our scientific culture, and failure to develop multiplicative reasoning has serious ramifications in the secondary school curriculum, as well as in everyday practical situations (Lamon & Lesh, 1992). Algebra, geometry, statistics, probability, calculus, biology, chemistry, and physics all require proportional reasoning abilities.

Proportional reasoning plays such a critical role in a student's mathematical development that it has been described as a watershed concept, a cornerstone of higher mathematics and a capstone of elementary concepts (Lesh, Post, & Behr, 1988). Further, the attainment of proportional reasoning is considered a milestone in a student's cognitive development (Cramer & Post, 1993). The importance of proportional reasoning is also stressed in the NCTM (1989) Standards.

Rational number ideas will eventually play a major role in the development of proportional reasoning abilities. The universal nature of rational number concepts in all of mathematics surely makes it one of the most important conceptual domains to be studied by students.

Vergnaud (1994) also points out that students, by the end of elementary school, have already been faced with some essential aspects of multiplicative structures. They have had to deal with different problems of multiplicative structures. They have had to deal with different problems of proportions, with different kinds of operations involving ratios and rates, and with different types of symbolism. At the same time, they have yet to build such high-level concepts as those of rational numbers, functions, and variables, dependence and independence. Furthermore, students will have to extend their knowledge of multiplicative structures to such difficult domains as geometry, probability, physics and so on.

However, students should extend their scope of valid intuitive knowledge to complex ratios and rates and to non-whole numbers. There are always strong epistemological obstacles to such an extension, such as the beliefs that one can not divide a number by a larger one, that multiplication makes bigger and that division makes smaller, and so on. It is important to find ways to improve students' understanding within this domain, since it is the foundation of all that is to come.

Shulman & Shulman (1994) concluded that experienced teachers understand that

when students leave whole numbers and enter the abyss of fractions, decimals, ratios, and percents, problems ensue. At that time, students who had previously succeeded begin to falter, their confidence wanes, failures increase, and teachers also despair. The problem is not only that children distrust their own mathematical intuitions, but also many elementary school teachers find their own mathematics understanding to be limited.

Research findings indicate that many concepts within the domain of multiplicative structures are not well taught nor are they well learned (Heller et al., 1990; Harel et al., 1988b; Simon & Blume, 1992). Research results also show that the mathematics structure of the multiplicative conceptual field is very complex and cognitively very demanding (e.g., Harel & Behr, 1989, 1990; Behr & Harel, 1990).

Research findings do show much about the way in which children are able to understand these concepts and the accompanying difficulties in teaching them (Post, 1989). It is time for such results to seriously impact school curricula. “They [research tasks] can function as instructional activities as well as assessment tools “(Cramer & Post, 1993b, p .407). These findings also offer a rich resource of assessing prospective teachers’ mathematical and pedagogical knowledge of multiplicative structures.

## METHODOLOGY

### **Evaluation Model: A Research-Based Approach**

The evaluation model, set up by a research-based approach, provided the foundation for assessing the prospective teachers’ mathematical and pedagogical content knowledge of multiplicative structures. The assessment problems were selected or adapted from assessment problems used previously for research. The literature on studying in the domain of multiplicative structures was reviewed from three perspectives: mathematical, psychological, and pedagogical analysis (see Figure 1: An Evaluation Framework for Assessing Teachers’ Knowledge of Multiplicative Structures).



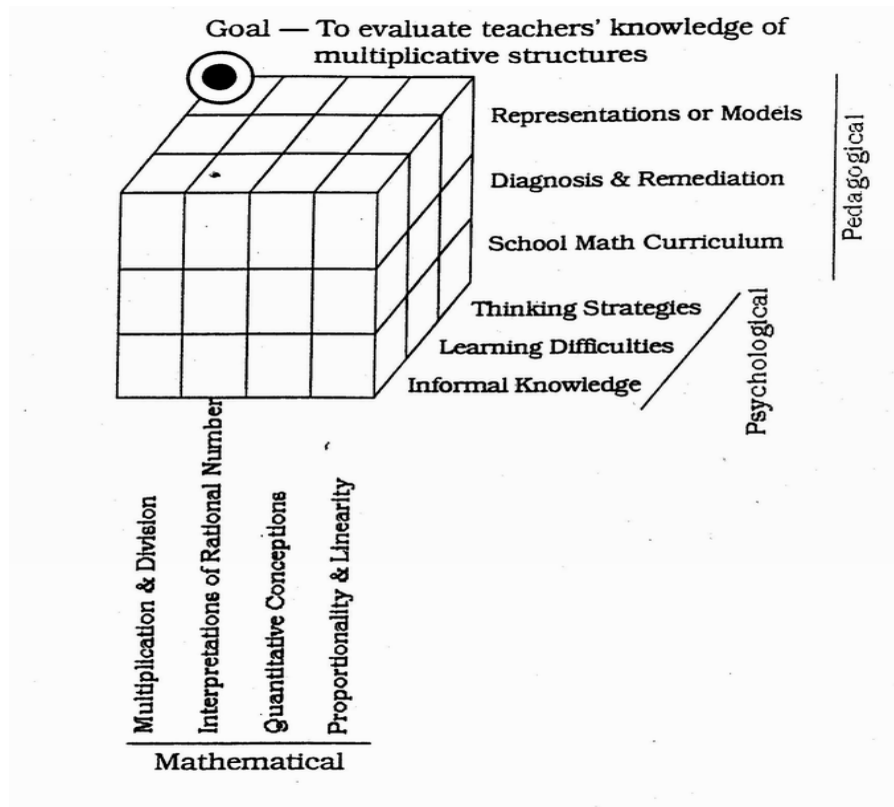


Figure 1: An Evaluation Framework for Assessing Teachers' Knowledge of Multiplicative Structures

The mathematical content knowledge of multiplicative structures consisted of four parts: multiplication and division, interpretations and relationships of rational numbers, quantitative conceptions, as well as proportionality and linearity.

The pedagogical content knowledge consisted of five parts: teaching representations, students' strategies, misconceptions and difficulties, remediation teaching, as well as school mathematics curriculum.

First, the researcher assessed the prospective teachers' ability to solve multiplication and division problems. Not only did the researcher assess what teachers did and did not know, but an attempt to discover their solution strategies was made. Were their solutions also influenced by Fischbein intuitive models?

Second, the researcher tried to understand how the prospective teachers

represented these concepts with a wide variety of models. Did they understand the relationships between these different interpretations or subconstructs of rational numbers?

Third, prospective teachers' quantitative conceptions of rational numbers were also investigated. Concepts such as the concepts of units, order and equivalence, operations and estimations are fundamental to the development of a viable quantitative conception of rational numbers. The researcher evaluated the prospective teachers' unit recognition, ability to order fractions, as well as estimation skills.

Fourth, the researcher examined the prospective teachers' proportional reasoning. In addition to solving numerical ratio and proportion problems - missing -valued and comparison problems, the prospective teachers' qualitative reasoning for solving questions of fraction order or equivalence, or the proportionality of two ratios, were also investigated. Finally, the prospective teachers were examined to discover whether they understood the mathematical characteristics of proportional situations that are necessary for teaching and learning ratio and proportions.

On the other hand, from the psychological and pedagogical perspectives, the researcher investigated whether the prospective teachers understood children's informal knowledge, learning difficulties and thinking strategies. The research questions were the following:

Could the prospective teachers represent appropriately their teaching methods using a wide variety of models? Would they acknowledge students' strategies? Could they diagnose their student misconceptions and learning difficulties? How would they do remediation teaching? Did they understand school mathematics curriculum?

Since prospective teachers might have had only a limited amount of teaching experience, they often would need to rely more on their own experience as students to address the issues in a particular teaching scenario.

Orton et al. (1995) propose that a teachers' synthesis of logical and psychological notions of rational number learning would be useful for building a model of teacher rationality. Results from researches could build a model of teacher rationality. Results from researches could be used to assess the extent to which teachers understand how multiplicative structures were learned. For instance, one could use

teachers' descriptions of how a hypothetical student would understand rational numbers to figure out a model of pedagogical reasoning (a la, Shulman, 1987). Teachers' reasoning would be evaluated in terms of "harmony" with the research base (Orton et al., 1995, p. 64). What is known about the learning of multiplicative concepts would then be used as a normative base for the assessment of teachers' pedagogical reasoning.

This evaluation method can be called a research-based approach. Cramer & Post (1993) argue that learning tasks devised in research studies can be a rich source of creative problem sets for classroom instruction and assessment. For the researcher, the widely various types of problems generated by research informs investigators of the different ways in which understanding can be assessed.

### **Assessment Problems: Design and Illustrations**

The assessment problems were selected according to the consideration of the range in Taiwanese elementary mathematics curriculum and the research data base which were available for evaluating the knowledge of multiplicative structures.

For assessing the prospective teachers' mathematical knowledge, the problems were selected to assess the prospective teachers' ability to solve multiplication and division problems, to understand how the prospective teachers tried to interpret meanings of rational numbers, to investigate the prospective teachers' quantitative conceptions of rational numbers, as well as to examine the prospective teachers' proportional reasoning.

For assessing the prospective pedagogical content knowledge, the problems were selected to examine the prospective teachers' teaching presentations, to investigate their acknowledgment of students' solution strategies, to investigate whether they could diagnose the students' misconceptions and learning difficulties, to examine their ability to do remediation teaching, and to investigate their acknowledgment of the elementary school mathematics curriculum.

The assessment problems were selected or adapted from assessment problems used previously for research. They were divided into two kinds of tests: a written test and an interview test, and had been administrated to investigate the prospective teachers' mathematical and pedagogical knowledge. The written test contained 39

items (W1-W39), whereas the interview test contained 10 items (I1-I10). The test items for interview were designed to take into account the complexity or to gain the subjects' insight and deeper understanding of the content knowledge. These assessment problems will be illustrated as follows, according to the evaluation model.

For examining the prospective teachers' mathematical knowledge of multiplicative structures, the first research problem was to investigate whether the teachers could solve the multiplication and division problems and how they solved the problems. The test items- W27, W28, and W29- were adapted from Harel et al. (1988). They were found to be the most difficult: less than 50% of the teachers provided a correct mathematical expression for getting a solution to the problems (Harel et al., 1994). They all violated Fischbein's intuitive rules, such as the divisor is greater than the dividend, the divisor is a non-whole number, the multiplier is a non-whole number less than one, and the product is smaller than the multiplicand.

The second research problem was used to understand how the subjects interpreted the meanings of the rational numbers. Would they represent these concepts with a wide variety of models? Would they understand the relationship between these different interpretations or subconstructs of rational numbers? The interview test item I1-1, which was adapted from Leu(1994), was used to examine their understanding of fractions. The six cards in item I1-2, adapted from Kerslake (1986), were used to force the subjects to recall the different meanings of rational numbers. The written test items-W7, W8, and W9- adapted from Lesh, Behr, & Post (1987), were used to examine the representations of rational numbers, whereas W5 (adapted from Barnett et al.,1994a) was used to test the meaning of the percentage.

The third research problem was to examine the prospective teachers' quantitative conceptions of rational numbers. The test items W18, W20, and W21 (adapted from Post & Cramer, 1996), as well as W19, are the "If X is  $n/m$ , of Y" type. The test item I7, adapted from Hart (1981) and Yang (1988), was also used to examine whether the subjects were able to identify a unit. The test item W14 examined the dense property of rational numbers. The test items W15, W16, and W17 were used to investigate the teacher's ability to order fractions. The test items W22, W23, W24, W25, and W26 (adapted from Post & Cramer, 1996) examined the estimation skills for the operations of fractions.

The fourth research problem was to investigate the prospective teachers' proportional reasoning. The test items W6, W30, and W32 (adapted from Lesh et al., 1987), as well as W31 (adapted from Kaput & West, 1994), were used to examine teachers' abilities to solve proportional situations. The interview test item I3 (adapted from Freudenthal, 1983; and Lamon & Lesh, 1992) is a real-life problem that is designed to elicit multiple solution strategies. The test items W1-W4, and W10-W13 (adapted from Post & Cramer, 1996) were used to examine qualitative reasoning on fractions and ratios. The item W33 adapted from Harel et al.'s (1992) block task is an effect problem. The item W34 examined the mathematical characteristics of proportional situations.

For investigating the prospective teachers' pedagogical content knowledge, the first research problem was used to examine their teaching presentations. The test item W38 (adapted from Barnett et al., 1994a) was designed to observe the subject's ability to explain  $0.20 = 0.2$ . In the test item W38, adapted from Kerslake (1986) and Yang, (1988), the question, "Do the subjects understand the use of number lines?", was answered. The interview test item I6 (adapted from Lesh et al., 1987; Post et al., 1991) measured whether or not the subjects could explain their thought processes they used in solving multiplication problems or proportion problems to their students. The item I10 (adapted from Leu, 1994) examined whether they understood the connection between the manipulation and the algorithm.

The second research problem was to examine the prospective teachers' acknowledgement of students' solution strategies. The test item I2-2 was used to predict how children who understood the concept of fractions ordered fractions. The item I8-2, adapted from Yang (1988) and Leu (1994), examined how children who had not been taught division with fractions solved word problem of division on fractions.

The third research problem was used to investigate whether the prospective teachers could diagnose the students' misconceptions and learning difficulties. The test item W35 (adapted from Kerslake, 1986; and Leu, 1994) examined the flexibility of their unit recognition, whereas W36a (adapted from Leu, 1994) and W38a (adapted from Lin, 1988) examined their representations of fractions. The test item W39a (adapted from Barnett et al., 1994a) examined their computation of algorithms. The interview test item I2-3 examined children's errors when ordering fractions. The test

item I4-1, adapted from Hirabayashi (1985) and Civil (1993), examined student's incorrect addition strategies in solving proportion problems.

The fourth research problem was used to investigate the prospective teacher's ability to do remediation teaching. The test items W36b, W39b, and I4-2 included these tasks.

The fifth and last research problem was to examine the prospective teachers' acknowledgement of elementary school mathematics curriculum. The test item W8, adapted from Yang (1988) and Leu (1994), was used to investigate whether the subjects would acknowledge the relationships between school curriculum and student's solution strategies. The test item I9-2, adapted from Yang (1988) and Leu (1994), examined whether the subjects' teaching sequence was consistent with the student's learning sequence.

## **Evaluation Procedures**

### **Subjects**

There were two steps in selecting the samples. In the first step, 417 out of all 457 seniors at one teachers' college in Taiwan attended the paper-and-pencil written test.

After completing the written tests, thirty seniors were selected from thirteen classes in eight different major programs. They volunteered or were recommended by the faculty in the teachers' college.

They were assessed with a semi-structured interview. Two students, almost always one female and one male, were sought from each class (except for the department of mathematics and science education) to participate in the interview. There were four students selected from each class in the department of mathematics and science education. The description of the participants in this study is presented in Table 1: Demographic Information of Participants.

**Table 1: Demographic Information of Participants**

| Group     | Math       | Education  |          |              | Art        |          |            |             | Total |
|-----------|------------|------------|----------|--------------|------------|----------|------------|-------------|-------|
| Written   | 68         | 227        |          |              | 122        |          |            |             | 417   |
|           | 37         | 111        |          |              | 51         |          |            |             | 199   |
| Male      |            |            |          |              |            |          |            |             |       |
|           | 31         | 116        |          |              | 71         |          |            |             | 218   |
| Female    |            |            |          |              |            |          |            |             |       |
| Interview | 8          | 14         |          |              | 8          |          |            |             | 30    |
| M/F       | 4/4        | 7/7        |          |              | 3/5        |          |            |             | 14/16 |
| Dept      | Math & Sci | Elementary | Language | Social Study | Visual Art | Music Ed | Special Ed | Physical Ed | 8     |
| Written   | 68         | 107        | 54       | 66           | 18         | 29       | 34         | 41          | 417   |
|           | 37         | 54         | 26       | 31           | 10         | 1        | 16         | 24          | 199   |
| Male      |            |            |          |              |            |          |            |             |       |
|           | 31         | 53         | 28       | 35           | 8          | 28       | 18         | 17          | 218   |
| Female    |            |            |          |              |            |          |            |             |       |
| Interview | 8          | 6          | 4        | 4            | 2          | 2        | 2          | 2           | 30    |
| M/F       | 4/4        | 3/3        | 2/2      | 2/2          | 0/2        | 0/2      | 2/0        | 1/1         | 14/16 |

**Administration of Assessment**

The written test was to assess prospective elementary teachers' mathematics content and pedagogical content knowledge. The written test was administered in about eighty minutes; however, there was no time limit. Calculators would not be permitted during the test. The test was given upon completion of the General Mathematics course and the Mathematics Teaching Methods course, the only two mathematics courses required by the teachers' college.

After finishing the written test, the researcher conducted the semi-structured interview. It took about eighty minutes. A semi-structured interview was investigated as a content-based assessment designed to take into account the complexity of teaching or to gain the subjects' insight and deeper understanding of the content knowledge.

The validity of the test items was evident from the previously researched database.

The Chinese version was also discussed and examined by the faculty in the department of mathematics education. The reliability of the written test was determined as .81 (coefficient Alpha), where  $N = 417$ .

### **Data Collection**

A pilot study was conducted for use and analysis of the instruments described previously. It was tested on two classes of juniors at the teachers' college.

The results obtained from the pilot study, such as the duration of test time, the usage of words in the test item, the answer types, the response types, etc., were used to revise the instrument. In order to collect research data, the first step was to administer the written test to all seniors.

After finishing the written test, the researcher conducted the semi-structured interview. All participants' responses to the interview were audiotaped and later translated to the protocols in the record sheet.

### **Data Analysis**

The written test and the interview test were both scored. The written test containing 39 items holds 77 grade points, whereas the interview test containing 10 items has 40 grade points.

The grade point for each test item was based on the contents and the formats of each test item. The grading system was based on the participants' answer types and response or solution strategies, according to the correctness of the answers and the rationality of explanations.

The mean scores of the different major group and department on the written test were used to analyze and to compare the prospective teachers' mathematical and pedagogical content knowledge between different major groups.

In the interview test, the scoring system converts the open-ended verbal responses of participants into a set of meaningful numerical scores. The scoring system evaluates the interview as a whole.

The test items were examined one by one. The answer types, solution strategies, and response categories were coded and analyzed.

The similarities and differences, and comparison of the results among three major groups or eight different departments were discussed and analyzed.



The two-way nested ANOVA was used to test whether there existed a significant difference of the prospective teachers' mathematical or pedagogical content knowledge between the different major groups. In the meantime, it also tested whether there existed a significant difference of the prospective teachers' mathematical or pedagogical content knowledge between the different departments within a major group. By the test of the two-way nested ANOVA, if there existed a significant effect of groups or departments, then follow-up t-tests were used to test all pairwise comparisons. The results would be regarded as the prospective teachers' mathematics knowledge profiles.

## RESULTS AND ANALYSES

### Mathematical Content Knowledge

The mathematical content knowledge of multiplicative structures consists of four parts: multiplication and division, interpretations and relationships of rational numbers, quantitative conceptions, as well as proportionality and linearity.

On the problems of multiplication and division, about 28% of the 417 prospective teachers had difficulty in solving multiplication and division problems when the numbers were expressed in decimal or fractional form. On average, about 9% of the participants applied wrong operations and found the incorrect answers. They might have been influenced by Fischbein intuitive models (Harel et al., 1994, p.365). There were four solution strategies in solving the multiplicative problems: non-proportion operations, ratio methods, unit-rate approaches, and building-down strategies. Twenty percent of the prospective teachers used two-step, unit-rate approaches to solve the partitive division problem, even though it is a one-step division problem. Those interviewees who used the ratio method might have given the correct answer, but gave inappropriate explanations. These explanations were totally procedural. There were 1.4% and 6% of the 417 prospective teachers who had computational errors in solving the multiplication and the partitive division problems, respectively. From these results, the prospective teacher's true computational abilities became apparent.

On the interpretations and relationships of rational numbers, nearly 10% of the 417 prospective teachers could not be sure that 100% is 1. More than 95% of the participants had the ability to perform translations between pictures and the written symbols (e.g.,  $2/3$ ) or the written language (e.g., two-thirds). On average, about 2% of the participants may have been affected by the perceptual distracters that were inherent in certain representations of rational numbers. A majority of the 30 prospective teachers (87%) used a continuous 'part-whole' model to explain the fraction  $3/5$ , whereas about 47% of the participants used a discrete 'part-whole' model. About 23% of the participants regarded  $3/5$  as a division ( $3 \div 5$ ). Nearly 17% of the participants thought of it as the decimal 0.6. About 13% of the participants regarded  $3/5$  as the ratio 3:5 or as a point on a number line. Only one participant expressed  $3/5$  as a percent (60%). About 60% of the 30 participants could recall one or two meanings of the fraction  $3/5$ ; and the remainder (40%) were able to express three or four meanings of the fraction  $3/5$ . Everyone could recall at least one meaning of the fraction  $3/5$ . No one could express five or more meanings of the fraction  $3/5$ . Only 20%-73% of the 30 prospective teachers could use conceptual meanings to illustrate the relationships between fractions and ratios, divisions, decimals, or percents, especially for the connection between fractions, ratios, and divisions. The remainder (a larger percentage 27%-80%) could not connect the relationships conceptually. On average, some participants (19%) used the values or the equations to express the relationships of the interpretations of rational numbers. Others (22%) connected the relationships with the symbols (" $\div$ " or " $:$ ") for fractions, ratios and divisions. Still others (28%) did not attempt to solve the problems. About one half of the 30 interviewees who refused to regard the ratio model (3 : 5) as the fraction ( $3/5$ ), were confused by the fraction-ratio relationship. No interviewee could illustrate clearly and completely the significant similarities and differences between a continuous model and a discrete model of rational number concepts. No interviewee could explicitly explain the discrete versus continuous distinction. Some participants also gave inadequate explanations to illustrate the number line and division models. In summary, the interpretations and relationships between meanings of fractions seemed to be not very well understood by those prospective teachers. A large percentage of those explanations relied on procedural approaches, rather than a pedagogically oriented understanding.

The quantitative conceptions include concepts of units, order fractions, and estimation skills. On the concepts of units, nearly 74% of the 417 prospective teachers could solve the problems of the types which were used to investigate the unit recognition (ex., If  $x$  is  $m/n$  of  $Y$ , find  $Y$ ). There were five major solution strategies for the above types of problems: the ratio algorithm, the division method, the algebra approach, the splitting-diagram strategy, and the building-up strategy. For the above problems of unit recognition, some unsuccessful solution attempts occurred when prospective teachers treated the fractional parts as a unit and showed  $m/n$ -th of it. Another misconception was in understanding the given fractional part as a unit fraction ( $1/n$ ) and then solving the problem. Moreover, some participants regarded one  $n$ -th as a unit. For the above problems of unit recognition in the discrete case, some participants thought that the elements in a unit should be integers. It seems that these prospective teachers thought that the word “number” (of circles or stars) implied only whole numbers. Almost all of 30 interviewees could identify a unit. However, they did not explain their reasons clearly. They were not aware of the lack of support for their judgment. Some prospective teachers wondered if there was a fraction like the fraction  $4.5/12$ . This might be a result of their definition of a fraction— “The definition of a fraction is that the numerator and the denominator should be natural numbers and greater than 1 ”.

On order fractions, about 87% of the 417 prospective teachers acknowledged the dense property of rational numbers. However, about 11% of the participants thought that there was more than 1, but finite fractions between  $1/2$  and  $1/4$ . Still about 2% of the prospective teachers thought that there was only one fraction between  $1/2$  and  $1/4$ . In the written test, more than 90% of the participants could correctly order fractions. About 7% of the prospective teachers could not order fractions. The prospective teachers employed six strategies to compare two fractions: (a) the common denominator or common numerator method, (b) the conversion of fractions into decimals, (c) the transitive strategy, (d) the residual strategy, (e) the inverse strategy, and (f) the formulas or rules. In addition to that, there was one more strategy used in the interview test. The prospective teachers compared two fractions with physical models, such as using fraction circles, drawing pictures or telling stories. When the prospective teachers compared two fractions, the biggest disadvantage was that they

just followed the procedures to solve the problem, but they did not concern themselves about what meanings or concepts were underlying the procedures. The prospective teachers were not willing to prove their formula. The worst strategy occurred when the prospective teachers applied incorrect math knowledge to solve problems.

On estimation skills, when estimating the results of additions or subtraction of fractions, on average, 60% of the prospective teachers did the estimation after they actually did the addition and subtraction of fractions. Only about 29% of the prospective teachers could do estimations before they did the computation. When estimating the results of multiplication or division of fractions, about 58% of the prospective teachers did the multiplication of fractions via estimations, whereas about 62% of the prospective teachers did the division of fractions via computations. About 5% of the 417 prospective teachers also applied incorrect mathematics knowledge to solve problems. For the real life problem - the measurement of the length of the room -, nine out of 21 prospective teachers got the best estimates with errors below 15%. Ten prospective teachers got estimates having the relative error below 50%. The remaining two prospective teachers had terrible estimates. Their relative errors were more than 100%. There were three kinds of estimation skills used to measure the length of the room: concrete material measurement, body measurement, and a perceptual basis. Using the concrete materials which were available at the time to make the estimation, was the best method. Second, it is also a successful way to make estimations by means of body measurement. However, at first one should acknowledge the sizes of the body. At worst, making estimations on a perceptual basis seemed to lead an inaccurate estimate.

Proportional reasoning includes the ability to solve proportional situations, the qualitative reasoning, and the understanding of the mathematical characteristics of proportional situations.

On proportional situations, almost 86% of the 417 prospective teachers could solve the proportional word problems correctly. The order of the missing value and the coordination of the measure spaces might have effected the level of difficulty for about 4% of the participants. There were four solution strategies used by these 417 prospective teachers in solving the missing-value proportional problems: the cross-product algorithm, the unit-rate strategy, the factor-of-change method, and the

fraction strategy. Being able to perform mechanical operations with proportions did not necessarily mean they understood the underlying ideas of proportional thinking. This is evident from the fact that they set up incorrect ratios and that they were not aware of the irrationality of their answers. Almost 90% of the 417 prospective teachers could correctly solve the proportional geometry figure problem. Only one prospective teacher used the additive strategy to solve this similar geometry problem. There were 5% of the prospective teachers who did not understand the meaning of similar figures. They confused similar figures with congruent figures and/or areas. For the problem of comparing the “orange flavor” of two orange juice mixes, about 84% of the participants could compare the “orange flavor” via equivalent ratios or fractions. When comparing the orange juice flavor problem, 9% of the participants were concerned about the state of the solution or the quality of the concentrate, and so could not make a decision. The participants performed poorly in the realistic proportional situation. Only two out of the 30 prospective teachers could apply proportional reasoning to estimate the height of the tower. At most, 63% of the participants used a measurement strategy. Some made his/her judgment purely on a perceptual basis. For the above real-life judgment problem, most explanations to support their answers were not mathematically acceptable, especially those based on their life experiences. Some prospective teachers were not aware that there were not exact measurements—all measurements were approximate.

For the problems of qualitative reasoning on fractions, more than 90% of the 417 prospective teachers could determine how the fraction would change for the three determinate cases. For the indeterminate case- where the numerator and denominator were both increased -, only 71% of the participants understood that this case was ambiguous. In the interview, about 63% of the 30 participants understood that the numerator and denominator could increase proportionally or non-proportionally, and gave three examples to show that the fraction might increase, decrease, or stay the same. The remainder could not give a complete explanation. One of the reasons might be a result of the disagreement in the meaning of “increase” (or “decrease”) between these prospective teachers. In contrast with the problems on fractions, for the qualitative reasoning on ratios, 91% of the participants could understand the indeterminate case: both the numerator and denominator were decreased. More than

83% of the participants could also determine how the ratio would change for the three determinate cases. Only 50% of the participants could correctly solve an effect problem, and make an induction. However, 20% of the participants insisted that this type of problem could not be solved, since the speeds were unknown.

On the problems of linearity, for a set of data, about 90% of the 417 prospective teachers acknowledged that it was a proportional situation. However, only 38% of the participants understood the mathematical characteristics of proportional situations. Only 3% of the participants learned to make sure that the graph of a proportional relationship forms a straight line through the origin. Another 32% of the participants' explanations were not sufficient or complete. About 8% of the participants did not understand the meaning of a proportional situation.

### **Pedagogical Content Knowledge**

The pedagogical content knowledge consists of five parts: teaching representations, students' strategies, misconceptions and difficulties, remediation teaching, as well as school mathematics curriculum.

On teaching presentations, the 417 prospective teachers' performance on the written test was not good enough to show their readiness for teaching. Teaching the topic like  $0.20=0.2$  but  $150 \neq 15$  is important in the 4<sup>th</sup>-grade mathematics curriculum (Ministry of Education, 1993). About 44% of the participants could depict student's incorrect usage of the shortcut and distinguish the difference between integers and decimals. Only 14% of the participants could apply the notation of place values to illustrate their explanations. Only 6% of the participants could make a connection between decimals and fractions. Only 5% of the participants were able to use the concrete materials to help their teaching. Number line interpretations were not easy for these prospective teachers. About 13% of the 417 prospective teachers were aware of the distinction between the position and the distance. About 55% of the participants acknowledged that the number line is an integration of visual and symbolic information. Some participants might have been confused between the measuring off of a fraction of the line and the placing of a point. When teaching the word problem of multiplication of a fraction, only 30% of the 30 prospective teachers could provide

adequate explanations to help children understand the operation. About 60% of the participants just set up the operation with very little explanation. Still two participants even used more complex approaches to teach this one-step multiplication problem. When using the manipulatives (ex., fraction circles) to teach the subtraction of fractions, only 30% of the 30 prospective teachers could distinguish the difference between the demonstration process and the computation procedure. The remainder (60%) could not connect the manipulation and the algorithm. The large percentage of the participants explained their thought processes procedurally, and the significant percentage of the participants did not answer the problems at all. In summary, a majority of the prospective teachers didn't seem to be able to represent appropriately their teaching methods using a wide variety of models.

On the problems of students' strategies, the 30 interviewees showed a good understanding of students' solution strategies. They acknowledged that children would apply at least five strategies to order fractions: physical models, formulas, common denominator or numerator methods, conversion to decimals approaches, and residual strategies. No one provided the transitive strategy. They also figured out that there were six solution strategies used by 5<sup>th</sup>-graders who had not learned the division of fractions in order to solve the word problem on the division of fractions. These strategies are: using repeated additions or subtractions, using diagrams, using equivalent fractions, and converting the kilometer into a meter.

On discovering misconceptions and difficulties, about 41% of the 417 prospective teachers could diagnose the students' misconceptions and learning difficulties. About 42% of the participants acknowledged the flexibility of the unit, whereas 22% of the participants insisted that a whole circle should be a unit. About 53% of the participants understood that the representation of figures affected the cognition of fractions. About 16% of the participants could point out that the different sequential processing of reading a fraction in English and Chinese caused students to have misconceptions about placing a fraction at the point on a number line. For instance, in England  $\frac{3}{5}$  is read as "three-fifths" (first read the numerator, next the denominator), whereas in Taiwan  $\frac{3}{5}$  is read as "wu fen zhu san" (first read the denominator "wu fen", next the numerator "san"). And 18% of the participants also understood that children had difficulties in regarding a fraction as a point on a number line. About 28% of the

participants could examine students' computation algorithms as a multiplication of mixed numbers. On average, about 70% of the 30 interviewees could use physical models to help children to order fractions, and 63% of the participants also emphasized the meaning of the denominator and the numerator. Ninety percent of the 30 interviewees could examine student's incorrect addition strategy in solving a proportion problems.

The findings showed the prospective teachers' lack of diagnostic teaching and remediation. For the remediation teaching of a representation of a fraction, only 6% of the 417 prospective teachers could use diagnostic teaching and 14% of the participants gave relational explanations to remediate the student's misconceptions. Nearly 31% of the participants gave the students the solution rules or let them do drills and practice. For the remediation of computation algorithms (ex., the multiplication of mixed numbers), only 2% of the 417 prospective teachers could do remediation teaching via diagram explanations. About 25% of the participant taught the meanings and algorithms of computation. Still 28% of the participants insisted on procedural computations. About 40% of the 30 interviewees could use proportions to remediate student's addition strategy.

On understanding the school mathematics curriculum, interviewing the 30 prospective teachers, 80% of subjects acknowledged that students could solve problems, even though they did not yet have the ability to do division with fractions. Twenty percent of the interviewees believed that the computation ability was fundamental to problem-solving. Some participants argued that the lack of the children's methods might be a result of too early, and too often abstract or symbolic teaching of algorithms. Less than 50% of the interviewees could judge student's difficulties rationally. Their teaching sequence was not consistent with students' learning sequence. About 53% of the interviewees judged the teaching situations subjectively rather than objectively.

### **Comparisons Between the Different Majors**

Based on the scores on the written test, the finding had indicated that the prospective teachers held a better understanding of mathematical content knowledge than that of pedagogical content knowledge. However, their overall performance was



not encouraging. The percentage correct was only 68%. The prospective teachers hold a severe lack of understanding of pedagogical content knowledge (the percent correct was 35%). They had a fair understanding of mathematical content knowledge (the percent correct was 80%). As a whole, the mathematics majors performed the best among all of the three groups. Except that the math majors did as well as the education majors on the test of interpretations and relationships of rational numbers, the mathematics majors held a better understanding than the other two majors on all of the tasks, no matter the mathematical or pedagogical content knowledge. The education majors did better than the art majors on all the tasks except for remediation teaching. The prospective teachers between three departments within the education major group did not perform significantly different on the overall content knowledge and the mathematical content knowledge. On the test of the pedagogical content knowledge, only the social studies majors performed better than the elementary education majors. The prospective teachers between four departments within the art major group had a significant difference on the mathematical content knowledge, especially for the quantitative conceptions, and proportionality and linearity. Within the art major group, regarding the understanding of content knowledge of multiplicative structures, the physical education majors were worse than the special education majors or the art education majors. The physical education majors showed less understanding of mathematical content knowledge than the special education majors and the art education majors. The special education majors had a better understanding of the mathematical content knowledge than the music education majors, especially regarding the quantitative conceptions. The special education majors also had a better understanding of proportionality and linearity than the art education majors.

## **DISCUSSION AND CONCLUSIONS**

The test was given upon completion of the General Mathematics course and the Mathematics Teaching Methods course. These participants were seniors in the teachers' college and had finished their course work and were ready to go on to their student teaching. The main concern regards what mathematical and pedagogical content knowledge they will bring into their coming student teaching.

These mathematics knowledge profiles indicated that the prospective teachers were not ready for teaching. Their level of pedagogical understanding was unacceptably low (35% correct). The mean score (80% correct) on the test of mathematical content knowledge was better but not completely satisfactory. The ideas examined in the test were fundamental for learning mathematics in the elementary school. However, they had difficulty in interpreting the relationships between the different meanings of rational numbers. A large percentage of their explanations relied on procedural approaches, rather than a pedagogically oriented understanding.

The prospective teachers might be able follow procedures (e.g., cross-product ratio method) to get the correct answer, but could not give appropriate explanations. Even when they set up incorrect ratios, or had computational errors, they were not aware of the irrationality of their answers.

The prospective teachers had many of the same misunderstandings and naïve conceptualizations, such as the additive strategy, that the researchers have identified in children. Some misunderstood the definitions of fractions, proportions, and similar figures. Even worse, they applied incorrect mathematics knowledge to solve the problems.

Indeed, the prospective teachers were not ready to teach. They often expressed their helplessness at the ideas of not knowing what to do when confronted with student's learning difficulties and misconceptions. An education major (E21) said, "I have never thought of that, did they think so? I would not try to solve that unless I wanted to teach it." Another education major (E22) even considered the option of neglecting to give feedback, as she said, "Now, they can't solve the problem, they can learn it after they grow up"; "Some kids can do that, and of course, some kids can't do that. Those who can solve the problem can do academic work; those who can't, let them go away," said she. This prospective teacher needs help. This brings up a concern about the prospective teachers' beliefs and attitudes toward the learning and teaching of mathematics.

The prospective teachers had little teaching experience; consequently, they lacked diagnostic teaching and remediation skills. Nearly 31% of the participants gave the students the solution rules or simply let them do drills and practice. They did not seem to be able to represent appropriately their teaching methods using a wide variety of

model.

Even though they advocated using concrete materials to help students' learning, the prospective teachers could not connect the "bridge" between the manipulation and the algorithm. This may be explained by their report that more than 76% of the interviewers had never used the fraction circles or other manipulative to learn fractions in their elementary schools.

The prospective teachers' lack of opportunities to learn to teach mathematics with manipulatives was also evident from the questionnaire survey during the interview. Only eight out of the thirty participants reported that they had manipulated fraction circles or Cuisenaire rods in the General Mathematics course or in the Mathematics Teaching Methods course, although these two manipulatives currently are often used to teach mathematics in the elementary schools.

Consequently, did the prospective teachers benefit from the teacher preparation program, even though the program required they take only four credit hours of mathematics and two credit hours of mathematics methods course?

Why did the mathematics majors perform the best among the three groups? Was it due to their strong mathematics background? The prospective teachers between different departments within the education major group had the same mathematical content knowledge, whereas they performed a significantly different on the pedagogical content knowledge. In contrast to that, there existed a significant difference on the mathematical content knowledge of the prospective teachers between different departments within the art major group, but there was no difference regarding the pedagogical content.

From the findings, it is not possible to fully identify what caused this situation. The effect of the program should be reexamined. However, the performance of the prospective teachers in the department of physical education showed the uneven quality of both the mathematics and methods preservice experiences.

What was the prospective teachers' perception of the program? Were they satisfied with the preparation program? A participant (L12) complained, "He [the instructor] taught Calculus which seemed to be important to the instructor, but for us, it is useless." Whereas, another (H12) applauded the instructor, who taught a variety of methods to teach elementary school mathematics. The relationship between the

learning of mathematics, the teaching mathematics, and the learning of how to teach mathematics is not clear. However, it is worth noting what a participant's (E21) comment, "I also wonder if the instructors in the teacher college are familiar with what's currently going on in the elementary school." It seemed that there appeared to be a gap between what these courses delivered and what the prospective teachers expected.

In this study, a model for assessing the prospective teachers' content knowledge was built up from three perspectives: mathematics, psychological, and pedagogical perspectives. The assessment results were used to develop the mathematical knowledge profiles of prospective teachers; there existed significant differences of prospective content knowledge between different majors. From this point of view, this assessment system could be used for the initial certification test to determine whether prospective teachers would enter their student teaching. These mathematics knowledge profiles of prospective teachers were developed to reflect their understanding of mathematical and pedagogical content knowledge.

What can be learned from this study? It is a matter of great concern for the researcher. The purpose of the study was not only to try to understand the status quo of the prospective teachers' mathematical and pedagogical content knowledge, but also to analyze the reasons, including attitudes to learning and teaching mathematics, behind their answers.

According to the mathematical knowledge profiles of the prospective teachers' in this study, some issues under discussion will come from the following questions: What degree of mathematical understanding do prospective teachers bring into their teaching practice? Why or why not did these prospective teachers gain a higher understanding of mathematics from the teachers' college? Could the evaluation model, used in this study for assessing prospective teachers' content knowledge, be used for the initial certification test in Taiwan?

In practice, more effort is needed to improve the assessment system for evaluating prospective teachers' content knowledge in order to certify teachers. Within the limits of time and cost, prospective teachers' knowledge must be assessed prior to certification to teach elementary school mathematics. In addition, much effort is needed to develop test items which are able to assess prospective teachers' pedagogical

content knowledge. Of course, the direct observation of prospective teachers' student teaching performance needs to be conducted in order to assess their pedagogical reasoning. However, this was difficult to conduct in this study, as well as in the initial certification test.

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## 職前教師乘法結構知識之研究

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### 摘 要

本研究乃在建立一個評鑑模式，用以評量職前教師的乘法結構知識及其教學知識。某一師範學院全體 417 位大四學生於畢業前，接受有關乘法結構知識的紙筆測驗，然後由該師範學院八個系中選出 30 位接受半結構之訪談。評量題目取材自以前相關研究問題加以修正，筆試之信度為  $\alpha = .81$ 。

研究顯示這些準教師尚未培育好去教數學，其有關乘法結構之教學知識的理解程度只有 35% 正確，而有乘法結構知識已達 80% 正確，但並非理想。他們缺乏診斷教學和補救教學知識，也不能利用各種合適的教學方式；他們既沒有意願去證明解題所需之公式，甚至使用不正確的數學知識。準教師解題解釋方式，大多數是程序性，而非教學啟示之瞭解。

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比較八個系的表現，無論乘法結構知識或其教學知識，都有顯著差異。整體而言，主修數理教育者表現最好，主修初等教育、語文教育和社會教育者表現優於主修特殊教育、體育教育、藝術教育和音樂教育者。

**關鍵詞：**職前教師、教師知識、乘法結構

